Uniform mean scalar gradient in grid turbulence: Asymptotic probability distribution of a passive scalar

X. D. Cai, E. E. O'Brien, and F. Ladeinde
Department of Mechanical Engineering, State University of New York at Stony Brook, Stony Brook, New York 11794-2300

(Received 14 February 1996; accepted 10 March 1996)

An asymptotic self-similar solution is obtained for the one-point probability density function (pdf) equation, of a passive scalar with uniform mean gradient in incompressible homogeneous turbulence. It is argued that the same solution should be a valid approximation when turbulence is generated in a high-quality wind tunnel. The asymptotic pdf shape is a unique function of the conditional expectation of the normalized scalar dissipation rate. The mean scalar gradient modifies the scalar pdf shape only if the conditional expected velocity component in the direction of the mean gradient is a nonlinear function of scalar fluctuation value. Experimental data from wind tunnel studies are consistent with the sign and scale of the changes produced by these nonlinearities. © *1996 American Institute of Physics.* [S1070-6631(96)01209-3]

A long-standing idealized problem in homogeneous turbulent transport of a dynamically passive scalar quantity is the case in which, in an unbounded domain, a uniform mean scalar gradient of the scalar is imposed initially in a direction perpendicular to the constant unidirectional mean velocity of the flow. The pedigree of this problem includes the original work of Corrsin,[1] who showed, analytically, that the mean scalar gradient remains constant as the scalar fluctuations develop and the scalar fluctuations are statistically homogeneous in planes orthogonal to the mean velocity. Subsequent experimental investigations in wind tunnels,[2,3] with the uniform mean scalar gradient generated by heating devices at an upstream location, have shown that Corrsin's predictions apply with good accuracy in the centerline region of the wind tunnel.

There is a well-known solution[4] for the asymptotic form of the pdf of the scalar field in the case when the imposed mean scalar gradient is zero. For the scalar normalized by its root mean square value, $\varphi_{\rm rms}$, its pdf is described by a single parameter, say χ, the expected value of the mean square scalar gradient conditioned on the scalar value. In this paper we obtain the corresponding asymptotic form of the scalar pdf when the mean scalar gradient is nonzero. It will be shown that this solution, which is exact when both the turbulence and scalar fields are statistically homogeneous, is described, in general, by χ and a second parameter, F_v, which is the expected value of the component of turbulent velocity in the direction of the mean scalar gradient conditioned by the scalar value. Here F_v plays a role only if it is a nonlinear function of scalar value.

We argue, from experimental evidence, that the same solution is a valid asymptotic approximation for the case of wind-tunnel turbulence when the mean scalar gradient is uniform, and that measured small nonlinear components of F_v are consistent with observations of the scalar skewness and flatness factors in wind-tunnel experiments.

Eswaran and Pope[5] found, in their numerical simulations of a passive scalar, φ, advected by a stationary homogeneous turbulence, that the probability distribution for the normalized scalar, $\varphi' = \varphi/\varphi_{\rm rms}$, arrived at an asymptotic distribution independent of the initial conditions. This result motivated Sinai and Yakhot[4] to search for the limiting probability distribution of a scalar, without mean gradient, in a random velocity field. Noting that there could be no limiting state for the time-evolving pdf of a non-normalized scalar, they obtained a pdf equation for the normalized scalar and showed that the limiting pdf is a unique function of the conditional expectation of the normalized scalar dissipation rate $\chi(\Phi)$, where Φ is a value of the random variable φ'. Jayesh and Warhaft[3] found, with some surprise, that the theory's prediction for the zero mean case fit very well their wind-tunnel experimental results for a scalar field with a mean gradient. This result suggests that the mean gradient, β, has no direct effect on the shape of the pdf.

The transport equation of a passive scalar with mean gradient β is[6]

$$\frac{\partial \varphi}{\partial t} + U\,\frac{\partial \varphi}{\partial x} + \nabla \cdot (\mathbf{v}\varphi) = D\ \nabla^2 \varphi - \beta v, \tag{1}$$

where φ is the scalar concentration fluctuation, U is the mean velocity (in the x direction), $\mathbf{v}$ is the fluctuating velocity and v is the component of $\mathbf{v}$ in the y direction, which is the direction of the uniform mean scalar gradient, and D is molecular diffusivity assumed to be constant. Yakhot[7] considered the limiting probability distribution of φ in the context of high Rayleigh number Bénard convection. His solution suggested that an exponential distribution of φ, at small φ, is a consequence of the plume-like structure of the velocity field of the Bénard problem at a high enough Rayleigh number. Our contribution deals with the possible effects of the mean gradient and the conditioned normalized velocity in wind-tunnel turbulence.

From (1),

$$\frac{\partial \varphi_{\rm rms}}{\partial t} + U\,\frac{\partial \varphi_{\rm rms}}{\partial x} + \frac{\nabla \cdot \langle \mathbf{v}\varphi^2\rangle}{2\varphi_{\rm rms}} = -\frac{1}{\varphi_{\rm rms}}\,(D\langle(\nabla\varphi)^2\rangle + \beta\langle v\varphi\rangle), \tag{2}$$

where we have omitted from the bracketed terms on the right-hand side of (2) a molecular diffusion term, $(D/2)\nabla^2\varphi_{\rm rms}^2$, which is zero in homogeneous fields and, in grid turbulence, is negligible[3] compared to the dissipation term $D\langle(\boldsymbol{\nabla}\varphi)^2\rangle$. Angular brackets denote an ensemble average.

Combining (1) and (2) and using the following definitions: $v'=v/u_{\rm rms}$, $\varphi'=\varphi/\varphi_{\rm rms}$, $\beta'=\beta u_{\rm rms}/\varphi_{\rm rms}$, $\epsilon'=\langle D(\boldsymbol{\nabla}\varphi/\varphi_{\rm rms})^2\rangle$, and $\rho=\langle v'\varphi'\rangle$, where $u_{\rm rms}$ is the turbulent velocity scale, we find

$$\frac{\partial\varphi'}{\partial t}+U\,\frac{\partial\varphi'}{\partial x}+\boldsymbol{\nabla}\cdot(\mathbf{v}\varphi')+\frac{\varphi'}{\varphi_{\rm rms}}\left(u\,\frac{\partial\varphi_{\rm rms}}{\partial x}-\frac{\boldsymbol{\nabla}\cdot\langle\mathbf{v}\varphi^2\rangle}{2\varphi_{\rm rms}}\right)$$
$$=D\,\frac{\nabla^2\varphi}{\varphi_{\rm rms}}-\beta'v'+\varphi'(\epsilon'+\beta'\rho). \tag{3}$$

Following the procedure of O'Brien[8] and Pope,[9] the equation for the pdf, $P(\Phi;x,t)$, can be written as

$$\frac{\partial P}{\partial t}+U\,\frac{\partial P}{\partial x}+\boldsymbol{\nabla}\cdot(\mathsf{F}P)-\frac{\partial}{\partial\Phi}\left[\left(\mathsf{F}\cdot\frac{\boldsymbol{\nabla}\varphi_{\rm rms}}{\varphi_{\rm rms}}-\frac{\boldsymbol{\nabla}\cdot\langle\mathbf{v}\varphi^2\rangle}{2\varphi_{\rm rms}^2}\right)\Phi P\right]$$
$$-D\,\nabla^2P=-\frac{\partial^2}{\partial\Phi^2}\left[D\left\langle\left(\frac{\boldsymbol{\nabla}\varphi}{\varphi_{\rm rms}}\right)^2\middle|\varphi'=\Phi\right\rangle P\right]$$
$$+\frac{\partial}{\partial\Phi}\left[\beta'\mathsf{F}_vP-(\epsilon'+\beta'\rho)\Phi P\right], \tag{4}$$

where $\mathsf{F}=\langle\mathbf{v}|\varphi'=\Phi\rangle$, $D\langle(\boldsymbol{\nabla}\varphi/\varphi_{\rm rms})^2|\varphi'=\Phi\rangle$, and $\mathsf{F}_v=\langle v'|\varphi'=\Phi\rangle$ are, respectively, the expected velocity fluctuation vector, scalar dissipation, and y-component normalized velocity fluctuation, all conditioned on the value of the normalized scalar fluctuation, Φ.

When the turbulence and scalar fluctuations are statistically homogeneous, it can be seen that all terms on the left-hand side of Eq. (4) are identically zero, except $\partial P/\partial t$, which is asymptotically zero in the limit of large times if a limit solution, $P(\Phi)$, exists. This is the same condition asserted by Sinai and Yakhot.[4] Only the terms on the right-hand side of (4) remain for the limiting solution. An exact solution is easily obtained for this case, but we postpone presenting it until the case of decaying turbulence in a wind tunnel is examined. The inhomogeneities in the streamwise direction then add some complexity to solving Eq. (4). However, an approximate solution, which has the same form as the exact solution in the statistically homogeneous case, can be argued from experimental evidence. Namely, the terms on the left-hand side of Eq. (4) are effectively negligible compared to the terms on the right-hand side of (4). A brief analysis of the terms on the left-hand side of (4) follows.

The first term on the left-hand side of (4) is zero since wind-tunnel turbulence is stationary. Experiments in wind tunnels[2,3] have shown,[6] as expected, that the third term on the left-hand side of (4), turbulent transport of probability, and the last term on the left-hand side of (4), molecular diffusion of probability, are both negligible compared to the production and dissipation terms on the right-hand side of (4). Similarly, experiments[10] have shown that $\boldsymbol{\nabla}\cdot\langle\mathbf{v}\varphi^2\rangle/2\varphi_{\rm rms}^2$, the turbulent transport of $\langle\varphi^2\rangle$, is negligible compared to ϵ', the rate of dissipation of $\langle\varphi^2\rangle$, which means that the second part of the fourth term on the left-hand side of (4) is also negligible. The first part of the fourth term can be assessed as follows. We note that, in wind-tunnel turbulence, $\mathsf{F}\cdot\boldsymbol{\nabla}\varphi_{\rm rms}/\varphi_{\rm rms}=\langle u|\varphi'=\Phi\rangle\cdot\partial\varphi_{\rm rms}/\varphi_{\rm rms}\,\partial x$, where u is the x component of the fluctuating velocity. It can be compared to the last term in (4). That is,

$$\frac{\langle u|\varphi'=\Phi\rangle\cdot\partial\varphi_{\rm rms}/\varphi_{\rm rms}\,\partial x}{\epsilon'}\sim\bigcirc\left(\frac{\langle u|\varphi'=\Phi\rangle}{u_{\rm rms}}\right). \tag{5}$$

Thus, considering that the unconditioned, fluctuating velocity is identically zero and the dependence of its conditioned streamwise component on Φ is negligible, if not zero, we argue that the first part of the fourth term on the left-hand side of (4) can safely be neglected. Furthermore, the measured asymptotic behavior of normalized scalar skewness and kurtosis,[3] which approach constants, suggested that the second term on left-hand side of (4), $U(\partial P/\partial x)$, is also effectively negligible compared to the production and dissipation terms in wind-tunnel turbulence as distance from the grid becomes large and the limiting shape of P becomes independent of x.

Thus, the exact statistically homogeneous case, and the approximate wind-tunnel representation, both satisfy the same equation (4), with the left-hand side equal to zero. On integrating (4), and assuming both $P(\Phi)$ and $\partial p/\partial\Phi$ decay rapidly enough as $|\Phi|\to\infty$ to set the constant of integration to zero, we find

$$\frac{\partial}{\partial\Phi}\left[\chi(\Phi)P(\Phi)\right]=\left\{\frac{\beta'}{\epsilon'}\cdot\left[\mathsf{F}_v(\Phi)-\Phi\rho\right]-\Phi\right\}P, \tag{6}$$

where

$$\chi=\frac{\langle D(\boldsymbol{\nabla}\varphi)^2|\varphi'=\Phi\rangle}{\langle D(\boldsymbol{\nabla}\varphi)^2\rangle}$$

is the normalized conditional dissipation.

The solution of (6) is

$$P(\Phi)=\frac{c}{\chi}\exp\left(-\int_{-\infty}^{\Phi}\frac{\Phi}{\chi}\,d\Phi\right)$$
$$\cdot\exp\left(-\int_{-\infty}^{\Phi}\frac{\beta'(\rho\Phi-\mathsf{F}_v)}{\epsilon'\chi}\,d\Phi\right). \tag{7}$$

As we noted above, (7) is a limiting solution in the sense of Sinai and Yakhot[4] if both the velocity and scalar fluctuating fields are statistically homogeneous. In the zero mean gradient case, $\beta'=0$, $P(\Phi)$ is determined by χ alone and (7) reduces to the solution obtained by Sinai and Yakhot.[4] Even if β' is nonzero, (7) becomes independent of β' if F_v is linear in Φ, say $\mathsf{F}_v=a\Phi$. This follows from the requirement that

$$\int_{-\infty}^{+\infty}\Phi\mathsf{F}_v(\Phi)P(\Phi)d\Phi=\rho,$$

when $a=\rho$ and $\mathsf{F}_v=\rho\Phi$. One situation in which this must occur is when v' and φ' are jointly Gaussian.

Evidence from experimental data[2] taken in wind-tunnel grid turbulence shows that F_v, while nearly a linear function of Φ in a uniform mean gradient, is not precisely so. Its dependence on Φ is obviously related to the statistical nature of the turbulence that carries the scalar, and to the initial state of the scalar field. There is evidence of both a small quadratic and a small cubic dependence on Φ. The latter affects the flatness factor, the former affects the skewness. Because both are small effects we treat them separately in the following analysis. They cannot negate each other.

Venkataramani and Chevray[2] measured F_v in grid generated turbulence. There is evidence in their results that F_v has a component with a cubic dependence on Φ (Ref. 2, Fig. 15). We estimate the cubic coefficient to be negative and approximately 1% of the linear coefficient. That is,

$$\mathsf{F}_v = a(\Phi + b\Phi^3), \tag{8}$$

where $b \sim -0.01$.

In this case,

$$\rho = \langle v'\varphi' \rangle = \int \mathsf{F}_v \varphi' P(\varphi') d\varphi' = a(1+bk),$$

where k is the kurtosis of Φ ($k=3$ for a Gaussian distribution and 6 for an exponential distribution). Hence

$$\mathsf{F}_v = \frac{\rho\Phi}{1+bk}(1+b\Phi^2), \tag{9}$$

and the second exponential factor in (7) becomes

$$\exp\left(-\frac{A}{4}\Phi^4 + \frac{kA}{2}\Phi^2\right), \tag{10}$$

where $A = -(\beta'/\epsilon'\chi)\rho[b/(1+bk)]$. For b small, A is linear in b and its magnitude is approximately $|b|$ since $\beta'\rho/\epsilon'\chi$ is of order unity due to the approximate balance of production and dissipation of scalar fluctuations in the flow. For a moderate value of Φ the factor (10) can then be approximated by $1+\frac{1}{4}|b|(\Phi^4 - 2k\Phi^2)$, which is greater than unity when $|\Phi| > \sqrt{2k}$ and less than unity around the origin of Φ. Consequently, a small increase in flatness factor can be expected when the conditioned expected velocity in the direction of the mean gradient has a cubic dependence on scalar fluctuating value, with a negative coefficient.

A dominant factor in determining the flatness factor of $P(\Phi)$ is likely to be χ, which, if the small-scale structure of the scalar field is isotropic, is necessary symmetric in Φ and modifies $P(\Phi)$ symmetrically. On the other hand, experimental data (Ref. 3, Fig. 11) support a slightly positively skewed probability density function $P(\Phi)$ under a uniform mean scalar gradient, while at the same time χ is symmetrically distributed for small Φ. Other wind-tunnel data,[2] without measurement of χ, shows the same asymmetry. A quadratic dependence of F_v on Φ can produce asymmetry in $P(\Phi)$, and measured data[2,3] support the prescription

$$\mathsf{F}_v = a(\Phi + c\Phi^2), \tag{11}$$

where $c \sim -0.01$ has been estimated from the experiments mentioned above.

Proceeding as before, we find

$$F_v = \rho\Phi(1+c\Phi)(1+cs)^{-1},$$

where s is the skewness of $P(\Phi)$, which is approximately $s=0.1$. In this case (10) becomes

$$\exp\left(-\frac{B}{3}\Phi^3 + \frac{sB}{2}\Phi^2\right), \tag{12}$$

where $B = (\beta'/\epsilon'\chi)\rho[c/(1+cs)]$, and (12) can be approximated, for small c and s and moderate $|\Phi|$, by

$$1+|c|\left(\frac{\Phi^3}{3} - \frac{s\Phi^2}{2}\right).$$

When Φ is negative, $P(\Phi)$ is reduced by this factor and, when Φ is positive and $\geqslant \frac{3}{2}s$, $P(\Phi)$ is enhanced. A small positive skewness can be expected when the conditioned expected velocity in the direction of the mean scalar gradient has a quadratic dependence on the scalar fluctuation value, with a negative coefficient.

In conclusion, we assert that there is an asymptotic solution (7) for the pdf of a scalar in homogeneous turbulence, which includes the case of a uniform mean scalar gradient. However, the scalar pdf is independent of the mean gradient if the expected value of the velocity component in the direction of the mean gradient, when conditioned on the scalar fluctuation value, is a linear function of such a value. Measurements in wind-tunnel flows with a uniform scalar gradient show evidence that the conditioned velocity has small quadratic and cubic nonlinearity components with respect to scalar fluctuation value. We have shown that the sign and magnitude of the measured changes in the scalar pdf in grid turbulence, with uniform scalar gradient, is consistent with solution (7). The derivation of (7) shows it to be a plausible solution for grid turbulence generated in a wind tunnel.

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