The flow of a power-law fluid in the near-wake of a flat plate

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The analysis of the near-wake flow downstream of a flat plate is reported in this paper for the case of a non-Newtonian (power-law) constitutive model. To our knowledge, the present paper is the first to address this problem, as previous work on near-wakes has been limited to the use of a Newtonian model. The motivation for this work comes from the biomedical engineering problem of blood flow around the bileaflet of a mechanical heart valve. In the present paper, the series method has been used to calculate the flow near the centerline of the wake, while an asymptotic method has been used for larger distances from the centerline. The effects of power-law inlet conditions on the wake flow are reported for various values of the power-law index n, within the range $0.7 \leq n \leq 1.3$. The present analysis has been successfully validated by comparing the results for $n=1$ to the near-wake results by Goldstein [Proc. Cambridge Philos. Soc. **26**, 1 (1930)]. We generalized the equations for arbitrary values of n, without any special considerations for $n=1$. Therefore, the accurate results observed for $n=1$ validate our procedure as a whole. The first major finding is that a fluid with smaller n develops faster downstream, such that decreasing n leads to monotonically increasing velocities compared to fluids with large n values. Another finding is that the non-Newtonian effects become more significant as the downstream distance increases. Finally, these effects tend to be more pronounced in the vicinity of the wake centerline compared to larger y locations. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2338825]

The investigation of the wake flow behind a flat plate started with the work of Goldstein,[1] who reported on the velocity distribution in the wake close to the trailing edge (near-wake). Tollmein[2] considered the problem of finding the first approximation to the asymptotic form of the two-dimensional wake far behind the flat plate (far-wake). The second-order approximation to the asymptotic form of the far-wake was developed by Goldstein[3] in his second paper. He matched the near-wake solution with the asymptotic solution numerically and reported on the origin of the coordinate in Tollmein's asymptotic solution. Using an integral method instead of Goldstein's numerical approach, Meksyn[4] obtained the motion in the near-wake. The previous studies focused on Newtonian fluids, with the exception of Liu and Wang,[5] who, in a recent paper, reported on a similarity solution for power-law fluids in the far-wake. The purpose of the present paper is to analyze the velocity distribution in the near-wake behind an infinite flat plate model of a mechanical heart valve (MHV) leaflet, assuming a steady laminar flow of a power-law fluid. The fact that human blood is a non-Newtonian fluid has been investigated,[6] and the power-law model is widely used for blood. However, no previous investigations have reported on the near-wake analysis for this type of fluid.

For small values of the transverse coordinate (y), that is, close to the centerline, the analytical procedure described in Goldstein[1] has been extended in this paper to be valid for non-Newtonian fluids. An asymptotic method is also developed for the non-Newtonian case and used for larger y values. The two solutions are consistent in the intermediate region.

Many previous studies have reported on the flow of a non-Newtonian fluid on a flat plate. However, we have developed the non-Newtonian inlet conditions imposed in the present paper for a general value of the non-Newtonian index n. The inlet power-law profile developed by Lemieux *et al.*[7] in an approximate analysis valid for $n \leq 1$ is also investigated.

Consider a two-dimensional flow of an incompressible power-law fluid over an infinite flat plate at zero incidence (Fig. 1). Let U_∞ represent the undisturbed velocity of the stream, l the length of the plate, y_1 the coordinate normal to the plate, x_1 the coordinate along the plate, and u_1 and v_1 the components of fluid velocity in x_1 and y_1 directions, respectively. The approximate nondimensional continuity and momentum equations determining a steady motion in the near-wake of the boundary layer flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n, \tag{2}$$

where the power-law assumption of $\tau_{xy}=K\left(\frac{\partial u_1}{\partial y_1}\right)^n$ has been evoked, $K/\rho=m, \rho$ is the mass density, m is a constant, and the molecular viscosity, μ, has been defined as $\mu=K\left(\frac{\partial u_1}{\partial y_1}\right)^{n-1}$.

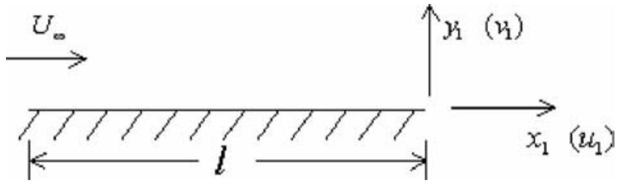

FIG. 1. A schematic of the coordinate system for the near-wake of a flat plate.

The generalized Reynolds number for a power-law fluid is defined as $\mathrm{Re}_L = U_\infty^{2-n}L^n/m$ and the following nondimensional scheme has been used:

$$x=\frac{x_1}{L},\quad y=\mathrm{Re}_L^{1/(n+1)}\frac{y_1}{L},\quad u=\frac{u_1}{U_\infty},\quad v=\mathrm{Re}_L^{1/(n+1)}\frac{v_1}{U_\infty}, \tag{3}$$

where $L=4l$.

These equations are subject to the following boundary conditions:

$$v=\frac{\partial u}{\partial y}=0 \quad at\ y=0 \quad \text{and} \quad u\to 1 \quad as\ y\to\infty. \tag{4}$$

The condition at the end of the plate ($x_1=0$), which serves as the inlet to the near-wake region, will be specified as the solution of the flow of a power-law fluid in the boundary layer. We develop a procedure for general values of n and, for validation purposes, examine power-law inlet conditions that are based on the approximate boundary-layer solutions by Lemieux *et al.*[7]

We introduce the following transformations for the normalized y coordinate η_b and stream function ψ_b:

$$\eta_b=\left(\frac{1}{n(n+1)}\right)^{1/(n+1)}x^{-1/(n+1)}y \tag{5}$$

and

$$\psi_b=\left(\frac{1}{n(n+1)}\right)^{-1/(n+1)}x^{1/(n+1)}\zeta, \tag{6}$$

to reduce the momentum equation to

$$\zeta(\zeta'')^{2-n}+\zeta'''=0 \tag{7}$$

and the boundary conditions to

$$\zeta'=0 \quad \text{and} \quad \zeta=0 \quad \text{at } \eta_b=0,$$
$$\zeta'=1 \quad \text{as } \eta_b\to\infty. \tag{8}$$

Note that this equation is consistent with Blasius' profile[8] when n is set equal to unity, but the focus here is on the case $n\neq 1$.

Equation (7) with the boundary conditions in Eq. (8) is solved numerically using the shooting method.[9] For small values of η_b (i.e., y), we can express the velocity distribution in the following power series form:

$$u=c_1(n)\,\eta_b+c_4(n)\,\eta_b^4+c_7(n)\,\eta_b^7. \tag{9}$$

Therefore, at the end of the flat plate, we have

$$u_i=a_1(n)y+a_4(n)y^4+a_7(n)y^7, \tag{10}$$

where the subscript "i" denotes "inlet" and the coefficients $a_i(n)$ can be easily calculated.

From the approximate analysis by Lemieux *et al.*,[7] which is valid for $n\leq 1$, the velocity profile at the wake inlet for power-law fluids can be written as

$$u_i=\mathrm{erf}[c(n)y]=1+\frac{2c(n)}{\sqrt{\pi}}\int_\infty^y e^{-[c(n)]^2y^2}dy, \tag{11}$$

where

$$c(n)=\left[\frac{0.982(n+1)^{1/2}}{n2^n\pi^{(2-n)/2}}\right]^{1/(n+1)}.$$

For convenience, we can also write Eq. (11) in the following form for small y:

$$u_i=a_1(n)y+a_4(n)y^4+a_7(n)y^7,$$

where the parameters a_1,a_4,a_7 are functions of n.

The near-wake equations for small η values can be written as follows:

$$\frac{n}{9^{n-1}}f_0'''+2f_0(f_0'')^{2-n}-(f_0')^2(f_0'')^{1-n}=0, \tag{12}$$

$$\frac{n}{9^{n-1}}f_3'''+\left[\frac{n(n-1)}{9^{n-1}}(f_0'')^{-1}f_0'''+2f_0(f_0'')^{1-n}\right]f_3''$$
$$-5f_0'(f_0'')^{1-n}f_3'+5(f_0'')^{2-n}f_3=0, \tag{13}$$

$$\frac{n}{9^{n-1}}f_6'''+\left[2f_0(f_0'')^{1-n}+\frac{n(n-1)}{9^{n-1}}(f_0'')^{-1}f_0'''\right]f_6''$$
$$-8f_0'(f_0'')^{1-n}f_6'+8(f_0'')^{2-n}f_6$$
$$=4(f_3')^2(f_0'')^{1-n}-5f_3f_3''(f_0'')^{1-n}-\frac{n(n-1)}{9^{n-1}}(f_0'')^{-1}f_3''f_3'''$$
$$-\frac{n(n-1)(n-2)}{2\times 9^{n-1}}(f_0'')^{-2}(f_3'')^2f_0''', \tag{14}$$

with the boundary conditions

$$f_i''(0)=f_i(0)=0,\quad i=0,3,6. \tag{15}$$

Equations (12)–(14) with the boundary conditions are also solved numerically using the shooting method.[9]

For large η, the stream function satisfies the equation

$$\psi=\psi_0+\xi\psi_1+\xi^2\frac{\psi_2}{2!}+\xi^3\frac{\psi_3}{3!}+\xi^4\frac{\psi_4}{4!}+\xi^5\frac{\psi_5}{5!}+\cdots, \tag{16}$$

where $\psi_i=\psi_i(y)$. Substitution of this expression into the momentum equation and extending the procedure in Goldstein[1] to the non-Newtonian case allows us to obtain the following expressions:

$$\psi_0'=\frac{1}{2}\zeta'=u_i,$$

$$\psi_1'=\frac{1}{2}A\zeta''=Au_i',$$

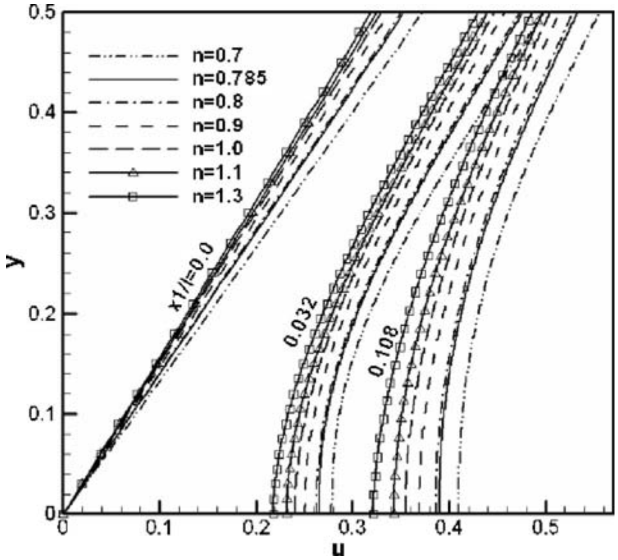

FIG. 2. Near-wake, small-y velocity distribution from the present analysis for general n. Results are shown for three values of the downstream distance x_1/l.

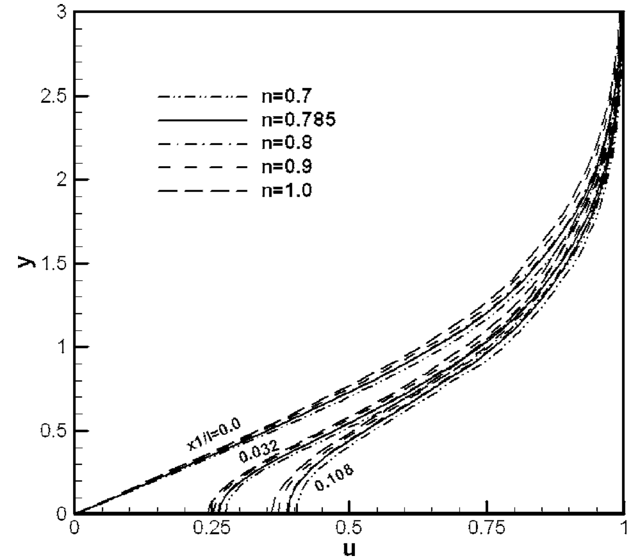

FIG. 3. Velocity distribution in the near-wake for the entire transverse (y) region, using various n values and Lemieux's power-law inlet conditions.

$$\frac{\psi_2'}{2!}=\frac{1}{2}\frac{A^2}{2!}\zeta'''=\frac{A^2}{2!}u_i'',$$

$$\frac{\psi_3'}{3!}=\frac{1}{2}\frac{A^3}{3!}\zeta^{(4)}-ny\zeta''=\frac{A^3}{3!}u_i'''-2nyu_i',$$

$$\frac{\psi_4'}{4!}=\frac{1}{2}\frac{A^4}{4!}\zeta^{(5)}-Any\zeta'''-Bn\zeta''=\frac{A^4}{4!}u_i^{(4)}-2Anyu_i''-2Bnu_i',$$

and

$$\frac{\psi_5'}{5!}=\frac{1}{2}\frac{A^5}{5!}\zeta^{(6)}-\frac{A^2}{2!}ny\zeta^{(4)}-ABn\zeta'''=\frac{A^5}{5!}u_i^{(5)}-2\frac{A^2}{2!}nyu_i'''-2ABnu_i'', \qquad (17)$$

where $A=3\delta_0, B=3\delta_0-1.5\delta_3$, and δ_0 and δ_3 are determined numerically.

The analyses have been carried out for general n values and investigated numerically for $0.7\leq n\leq 1.3$, which includes both pseudoplastic and dilatant fluids. The procedure presented in this paper was successfully validated for the $n=1$ case, using the results in Goldstein[1] (not shown). We have generalized the procedure for arbitrary values of n, without any special modifications for $n=1$. Therefore, accurate results for $n=1$ imply that the procedure as a whole is correct. The velocity distribution in the near-wake using the inlet conditions we developed for the case with general n is shown in Fig. 2 for $0.7\leq n\leq 1.3$. Note in this figure that, for $x_1/l=0.0$, the velocity starts out at zero at $y=0$ and increases linearly with y. At larger y values (see Fig. 3, obtained from Lemieux's inlet profile), the gradient decreases, eventually remaining constant at the free-stream value as y becomes large. These features are essentially those of flow "over" a flat plate, in which the effects of different n values are shown. As we move away from $x=0$ into the wake region in Fig. 2, the centerline velocity has finite values, with a zero gradient at $y=0$. Thus, whereas we see a boundary-layer profile at $x=0$, the distribution at downstream x values is evidently that of a wake. For the three x stations shown in Fig. 2, we observe lower velocities with increasing n values.

The results in Figs. 2 and 3 can be explained quite easily by examining the profiles (at $x_1/l=0.032$) (not shown) of the velocity gradient, du/dy, viscosity, $\mu/K\equiv(du/dy)^{n-1}$, and the shear stress, $\tau_{xy}/K\equiv(du/dy)^n$. Most of the positive velocity gradient at $x_1/l=0.032$ is located in $0\leqslant y\leqslant 0.4$, followed by decreasing gradients at large y. We also see relatively large viscosity values close to the centerline $y=0$ when n is small (not shown), with an exponential decrease with y up until approximately $y=0.4$. At this point, the fluid becomes essentially Newtonian as the shear rates (du/dy) approach constant values for all values of n. The equilibrium (Newtonian) viscosity values at large y are different for different n, although this observation is of little dynamic significance. The normalized shear stress shows decreasing values with increasing n, consistent with the viscosity and shear rate distributions.

The effects of n on the velocity distribution at various locations ($x_1/l=0$, 0.032, and 0.108) are also shown in Fig. 2 for general n. The most dramatic effects occur at small values of y, where velocity decreases monotonically with increasing n. The differences in the results for the various n values decrease with increasing y, until the velocity converges to $u=1$ at infinite y (not shown). Note that the trends above also apply to cases in which $n>1$. The case $n=0.785$ is of interest and has been included in the plots, as this value represents the most commonly used index for human blood.[6] Note that an excellent agreement was observed between the solutions using general n and those based on Lemieux's. However, the two results are not exactly the same, because of the approximations used in Lemieux's procedure. In fact, the Lemieux approximation does not give the same coefficients and does not reduce to the Blasius solution at $n=1$.

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